

COISOTROPIC RIGIDITY AND C^0 -SYMPLECTIC GEOMETRY

VINCENT HUMILIÈRE, RÉMI LECLERCQ, SOBHAN SEYFADDINI

ABSTRACT. We prove that symplectic homeomorphisms, in the sense of the celebrated Gromov–Eliashberg Theorem, preserve coisotropic submanifolds and their characteristic foliations. This result generalizes the Gromov–Eliashberg Theorem and demonstrates that previous rigidity results (on Lagrangians by Laudenbach–Sikorav, and on characteristics of hypersurfaces by Opshtein) are manifestations of a single rigidity phenomenon. To prove the above, we establish a C^0 -dynamical property of coisotropic submanifolds which generalizes a foundational theorem in C^0 -Hamiltonian dynamics: Uniqueness of generators for continuous analogs of Hamiltonian flows.

1. INTRODUCTION AND MAIN RESULTS

A submanifold C of a symplectic manifold (M, ω) is called coisotropic if for all $p \in C$, $(T_p C)^\omega \subset T_p C$ where $(T_p C)^\omega$ denotes the symplectic orthogonal of $T_p C$. For instance, hypersurfaces and Lagrangians are coisotropic. A coisotropic submanifold carries a natural foliation \mathcal{F} which integrates the distribution $(TC)^\omega$; \mathcal{F} is called the characteristic foliation of C . Coisotropic submanifolds and their characteristic foliations have been studied extensively in symplectic topology. The various rigidity properties that they exhibit have been of particular interest. For example, in [7] Ginzburg initiated a program for studying rigidity of coisotropic intersections. In this paper, we prove that coisotropic submanifolds, along with their characteristic foliations, are C^0 -rigid in the spirit of the Gromov–Eliashberg Theorem.

The Gromov–Eliashberg Theorem states that a diffeomorphism which is a C^0 -limit of symplectomorphisms is itself symplectic. Motivated by this, symplectic homeomorphisms are defined as C^0 -limits of symplectomorphisms (see Definition 11). Area preserving homeomorphisms, and their products, are examples of symplectic homeomorphisms. Here is our main result.

Theorem 1. *Let C be a smooth coisotropic submanifold of a symplectic manifold (M, ω) . Let U be an open subset of M and $\theta: U \rightarrow V$ be a symplectic homeomorphism. If $\theta(C \cap U)$ is smooth, then it is coisotropic. Furthermore, θ maps the characteristic foliation of $C \cap U$ to that of $\theta(C \cap U)$.*

Date: May 7, 2013.

2010 Mathematics Subject Classification. Primary 53D40; Secondary 37J05.

Key words and phrases. symplectic manifolds, coisotropic submanifolds, characteristic foliation, C^0 -symplectic topology, spectral invariants.

An important feature of the above theorem is its locality: C is not assumed to be necessarily closed and θ is not necessarily globally defined. Here is an immediate, but surprising, consequence of Theorem 1.

Corollary 2. *If the image of a coisotropic submanifold via a symplectic homeomorphism is smooth, then so is the image of its characteristic foliation.*

Theorem 1 uncovers a link between two previous rigidity results and demonstrates that they are in fact extreme cases of a single rigidity phenomenon.

One extreme case, where C is a hypersurface, was established by Opshtein [22]. Clearly, in this case, the interesting part is the assertion on rigidity of characteristics, as the first assertion is trivially true.

Lagrangians constitute the other extreme case. When C is Lagrangian, its characteristic foliation consists of one leaf, C itself. In this case the theorem reads: *If θ is a symplectic homeomorphism and $\theta(C)$ is smooth, then $\theta(C)$ is Lagrangian.* In [11], Laudénbach–Sikorav proved a similar result: *Let L be a closed manifold and ι_k denote a sequence of Lagrangian embeddings $L \rightarrow (M, \omega)$ which C^0 -converges to an embedding ι . If $\iota(L)$ is smooth, then (under some technical assumptions) $\iota(L)$ is Lagrangian.* On one hand, their result only requires convergence of embeddings while Theorem 1 requires convergence of symplectomorphisms. On the other hand, Theorem 1 is local: It does not require the Lagrangian nor the symplectic manifold to be closed.

The above discussion raises the following question.

QUESTION. What can one say about C^0 -limits of coisotropic embeddings and their characteristic foliations?

We would like to point out that Theorem 1 is a coisotropic generalization of the Gromov–Eliashberg Theorem. Indeed, it implies that if the graph of a symplectic homeomorphism is smooth, then it is Lagrangian.

Theorem 1 allows us to define C^0 -coisotropic submanifolds. Below, we assume that \mathbb{R}^{2n} is equipped with the standard symplectic structure. Recall that every coisotropic submanifold of codimension k is locally symplectomorphic to

$$\mathcal{C}_0 = \{(x_1, \dots, x_n, y_1, \dots, y_n) \mid (y_{n-k+1}, \dots, y_n) = (0, \dots, 0)\} \subset \mathbb{R}^{2n}.$$

See [14, Proposition 13.7] and [8].

Definition 3. *A codimension- k C^0 -submanifold C of a symplectic manifold (M, ω) is C^0 -coisotropic if around each point $p \in C$ there exists a C^0 -coisotropic chart, that is, a pair (U, θ) with U an open neighborhood of p and $\theta: U \rightarrow V \subset \mathbb{R}^{2n}$ a symplectic homeomorphism, such that $\theta(C \cap U) = \mathcal{C}_0 \cap V$.*

A codimension- n C^0 -coisotropic submanifold is called a C^0 -Lagrangian.

EXAMPLE. Graphs of symplectic homeomorphisms are C^0 -Lagrangians. Graphs of differentials of C^1 functions and, more generally, graphs of C^0

1-forms, closed in the sense of distributions, provide a family of non trivial examples, see Appendix B for a proof.

Conversely, we could ask whether every continuous 1-form whose graph is a C^0 -Lagrangian is closed in the sense of distributions. An affirmative answer in a particular case appears in Viterbo [25, Corollary 22].

We will denote by \mathcal{F}_0 the characteristic foliation of \mathcal{C}_0 and by

$$\mathcal{F}_0(p) = \{(x_1, \dots, x_{n-k}, t_{n-k+1}, \dots, t_n, y_1, \dots, y_{n-k+1}, 0, \dots, 0) \mid (t_{n-k+1}, \dots, t_n) \in \mathbb{R}^k\}$$

the leaf of \mathcal{F}_0 passing through $p = (x_1, \dots, x_n, y_1, \dots, y_{n-k}, 0, \dots, 0)$.

As a consequence of Theorem 1, C^0 -coisotropic submanifolds carry a (C^0 -)characteristic foliation in the following sense.

Corollary 4. *Any C^0 -coisotropic submanifold C admits a unique C^0 -foliation \mathcal{F} which is mapped to \mathcal{F}_0 by any C^0 -coisotropic chart.*

EXAMPLE. If $C = \theta(C')$, with C' a smooth coisotropic submanifold and θ a symplectic homeomorphism, then $\mathcal{F} = \theta(\mathcal{F}')$ where \mathcal{F}' is the characteristic foliation of C' .

One may wonder if every topological hypersurface is C^0 -coisotropic. It is possible to show, via an application of Corollary 4, that the boundary of the standard cube in \mathbb{R}^4 does not possess a C^0 -characteristic foliation, and hence, it is not C^0 -coisotropic. This, in particular, implies that it is not possible to map the boundary of the sphere to that of the cube by symplectic homeomorphisms.

Corollary 5. *A smooth C^0 -coisotropic submanifold is coisotropic and the natural C^0 -foliation it carries coincides with its characteristic foliation.*

As we shall see, the proof of Theorem 1 relies on dynamical properties of coisotropic submanifolds. In particular, we use C^0 -Hamiltonian dynamics as defined by Müller and Oh [21]. To the best of our knowledge, this is the first extrinsic application of this recent, yet promising, theory.

Following [21], we call a path of homeomorphisms ϕ^t a hameotopy if there exists a sequence of smooth Hamiltonian functions H_k such that the isotopies $\phi_{H_k}^t$ C^0 -converge to ϕ and the Hamiltonians H_k C^0 -converge to a continuous function H (see Definition 12). Then, H is said to generate the hameotopy ϕ^t , and to emphasize this we write ϕ_H^t ; the set of such generators will be denoted C_{Ham}^0 . A foundational result of C^0 -Hamiltonian dynamics is the uniqueness of generators Theorem (see [25, 3]) which states that the trivial hameotopy, $\phi^t = \text{Id}$, can only be generated by those functions in C_{Ham}^0 which solely depend on time (see also Corollary 7 below).

Recall the following dynamical property of smooth coisotropic submanifolds: Let $H \in C^\infty(\mathbb{S}^1 \times M)$, $H|_C$ is a function of time if and only if ϕ_H

(preserves C and) flows along the characteristic foliation of C . By flowing along characteristics we mean that for any point $p \in C$ and any time $t \geq 0$, $\phi_H^t(p) \in \mathcal{F}(p)$, where $\mathcal{F}(p)$ stands for the characteristic leaf through p . The following result, which plays an important role in the proof of Theorem 1, is a C^0 -analog of the above.

Theorem 6. *Denote by C a connected and closed C^0 -coisotropic submanifold of a symplectic manifold (M, ω) . Let $H \in C_{\text{Ham}}^0$ with induced hamotopy ϕ_H . The restriction of H to C is a function of time if and only if ϕ_H preserves C and flows along the leaves of its characteristic foliation.*

This result answers a question raised by Buhovsky and Opshtein who asked if the above holds in the particular case where C is a smooth hypersurface. It also drastically generalizes the aforementioned uniqueness of generators Theorem. Indeed, if C is taken to be M , then the characteristic foliation consists of the points of M and the theorem follows immediately:

Corollary 7. *$H \in C_{\text{Ham}}^0$ is a function of time if and only if $\phi_H^t = \text{Id}$.*

When C is a C^0 -Lagrangian, Theorem 6 states that: *The restriction of H to L is a function of time if and only if $\phi_H^t(L) = L$ for all t .* Interestingly enough, the general case of both Theorems 1 and 6 will be essentially deduced from the a priori particular case of Lagrangians. This is, most probably, a C^0 manifestation of Weinstein's creed: "Everything is a C^0 -Lagrangian submanifold!"

The results of this paper establish C^0 -rigidity of coisotropic submanifolds together with their characteristic foliations. It would be interesting to see if isotropic or symplectic submanifolds exhibit similar rigidity properties: If a smooth submanifold is the image of an isotropic (respectively symplectic) submanifold under a symplectic homeomorphism, is it isotropic (respectively symplectic)? Note that if in these questions one considers, instead of symplectic homeomorphisms, C^0 -limits of isotropic (respectively symplectic) embeddings then Gromov's results on the h -principle provide negative answers in general. In short, isotropic and symplectic embeddings are not C^0 -rigid. (See [9, Section 3.4.2], or [5, Theorems 12.1.1 and 12.4.1].)

Main tools: Lagrangian spectral invariants. In order to prove the main results, we use the theory of Lagrangian spectral invariants. One consequence of this theory is the existence of the spectral distance γ on the space of Lagrangians Hamiltonian isotopic to the 0-section in cotangent bundles introduced by Viterbo in [24].¹

More precisely, we establish inequalities comparing γ to a capacity recently defined by Lisi–Rieser [15]. This capacity, which we denote by c_{LR} , is a relative (to a fixed Lagrangian) version of the Hofer–Zehnder capacity. We

¹One of the main features of γ is that it is bounded from above by Hofer's distance on Lagrangians. In particular, Lemmas 8 and 9 also hold with γ replaced by Hofer's distance.

will now define c_{LR} . Fix a Lagrangian L . Recall that a Hamiltonian chord of a Hamiltonian H , of length T , is a path $x: [0, T] \rightarrow M$ such that $x(0), x(T) \in L$ and for all $t \in [0, T]$, $\dot{x}(t) = X_H^t(x(t))$. A Hamiltonian is said to be L -slow if all of its Hamiltonian chords of length at most 1 are constant. We denote by $\mathcal{H}(U)$ the set of admissible Hamiltonians, that is, smooth time-independent functions with compact support included in U , which are non-negative and reach their maximum at a point of L . For an open set U which intersects L , the relative capacity of U with respect to L is defined as

$$c_{\text{LR}}(U; L) = \sup\{\max f \mid f \in \mathcal{H}(U) \text{ } L\text{-slow}\}.$$

For instance, if B is the ball of radius r in \mathbb{R}^{2n} and $\mathcal{L}_0 = \mathbb{R}^n \times \{0\}$, then $c_{\text{LR}}(B; \mathcal{L}_0) = \frac{\pi r^2}{2}$.

In what follows, we denote by L_0 the 0-section of T^*L . The first energy-capacity inequality used in this paper is the following:

Lemma 8. *Let L be a smooth closed manifold. Let U_- and U_+ be open subsets of T^*L , so that $U_{\pm} \cap L_0 \neq \emptyset$. Assume that $c_{\text{LR}}(U_-; L_0) = c_{\text{LR}}(U_+; L_0)$ and let C_{\pm} be real numbers such that $C_{\pm} > c_{\text{LR}}(U_{\pm}; L_0)$. If a compactly supported Hamiltonian H satisfies $H|_{U_{\pm}} = \pm C_{\pm}$, then $\gamma(\phi_H^1(L_0), L_0) \geq c_{\text{LR}}(U_{\pm}; L_0)$.*

This is the Lagrangian analog of the energy-capacity inequality proven for the Hamiltonian spectral distance in [10, Corollary 12]. Then, as in [10], we will derive a similar inequality for Hamiltonians (not necessarily constant but) with controlled oscillations on U_{\pm} , see Corollary 15.

Lemma 8 can also be established on compact manifolds for weakly exact Lagrangians via Leclercq [12] and for monotone Lagrangians via Leclercq–Zapolsky [13].

The second energy-capacity inequality is due to Lisi–Rieser [15]. This is a relative version of the standard energy-capacity inequality, see for example Viterbo [24].

Lemma 9. *Let L be a smooth closed manifold. Suppose that U is an open subset of T^*L , with $L_0 \cap U \neq \emptyset$. Assume that L' is a Lagrangian Hamiltonian isotopic to L_0 such that $L' \cap U = \emptyset$. Then $\gamma(L', L_0) \geq c_{\text{LR}}(U; L_0)$.*

A special case of this specific inequality appears in Barraud–Cornea [1] and Charette [4]. A similar inequality is worked out in Borman–McLean [2].

Finally, we will need an inequality which provides an upper bound for the spectral distance. Let g denote a Riemannian metric on a closed manifold L and denote by $T_r^*L = \{(q, p) \in T^*L \mid \|p\|_g \leq r\}$ the cotangent ball bundle of radius r . Suppose that $\phi_H^t(L_0) \subset T_r^*L$ for all $t \in [0, 1]$. Viterbo has conjectured [26] that there exists a constant $C > 0$, depending on g , such that $\gamma(\phi_H^1(L_0), L_0) \leq Cr$. This conjecture has many important ramifications; see

[17, 26]. Lemma 10 below is a special case of Viterbo's conjecture; a more precise version of the lemma appears in [20, Theorem 9.7].

Lemma 10. *Let L be a smooth closed manifold, \mathcal{V} a proper open subset of L , and $V = \pi^{-1}(\mathcal{V}) \subset T^*L$, where $\pi: T^*L \rightarrow L$ is the standard projection. There exists $C > 0$, depending on the set V , such that: For all $r > 0$, if H is a smooth, compactly supported Hamiltonian on T^*L such that $H|_V = 0$, and $\phi_H^t(L_0) \subset T_r^*L$ for all $t \in [0, 1]$ then $\gamma(\phi_H^1(L_0), L_0) \leq Cr$.*

Organization of the paper. In Section 2, we review the preliminaries on C^0 -Hamiltonian dynamics and Lagrangian spectral invariants. In Section 3, we prove energy-capacity inequalities (Lemmas 8 and 9) as well as the upper bound on the spectral distance (Lemma 10). In Section 4, we use these inequalities in order to prove localized versions of Theorem 6 in the special case of Lagrangians. In Section 5, we prove Theorems 1 and 6 and Corollaries 4 and 5 using the results of Section 4.

In Appendix A, we provide relatively simple, and hopefully enlightening, proofs of Theorems 1 and 6 in the special case of closed Lagrangians in cotangent bundles. We hope that this appendix will give the reader an idea of the proofs of the main results while avoiding the technicalities of Sections 4 and 5. In Appendix B, we prove that the graph of a closed C^0 1-form is a C^0 -Lagrangian.

Acknowledgements. We thank Samuel Lisi and Tony Rieser for sharing their work with us before it was completed and for related discussions. The inspiration for this paper came partly from an unpublished work by Lev Buhovsky and Emmanuel Opshtein, whom we also thank. We are especially grateful to Emmanuel Opshtein for generously sharing his ideas and insights with us through many stimulating discussions. We are also grateful to Alan Weinstein for interesting questions and suggestions.

This work is partially supported by the grant ANR-11-JS01-010-01.

2. PRELIMINARIES

2.1. Symplectic and Hamiltonian homeomorphisms. In this section we give precise definitions for symplectic and Hamiltonian homeomorphisms and recall a few basic properties of the theory of continuous Hamiltonian dynamics developed by Müller and Oh [21].

Given two manifolds M_1, M_2 , a compact subset $K \subset M_1$, a Riemannian distance d on M_2 , and two maps $f, g: M_1 \rightarrow M_2$, we denote

$$d_K(f, g) = \sup_{x \in K} d(f(x), g(x)).$$

We say that a sequence of maps $f_i: M_1 \rightarrow M_2$ C^0 -converges to some map $f: M_1 \rightarrow M_2$, if for every compact subset $K \subset M_1$, the sequence $d_K(f_i, f)$ converges to 0. This notion does not depend on the choice of the Riemannian metric.

Definition 11. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A continuous map $\theta: U \rightarrow M_2$, where $U \subset M_1$ is open, is called *symplectic* if it is the C^0 -limit of a sequence of symplectic diffeomorphisms $\theta_i: U \rightarrow \theta_i(U)$.

Let $U_1 \subset M_1$ and $U_2 \subset M_2$ be open subsets. If a homeomorphism $\theta: U_1 \rightarrow U_2$ and its inverse θ^{-1} are both symplectic maps, we call θ a *symplectic homeomorphism*.

Clearly, if θ is a symplectic homeomorphism, so is θ^{-1} . By the Gromov–Eliashberg Theorem a symplectic homeomorphism which is smooth is a symplectic diffeomorphism. We now turn to the definition of Hamiltonian homeomorphisms (called *hameomorphisms*) introduced by Müller and Oh [21].

Definition 12. Let (M, ω) be a symplectic manifold and $I \subset \mathbb{R}$ an interval. An isotopy $(\phi^t)_{t \in I}$ is called a *hameotopy* if there exist a compact subset $K \subset M$ and a sequence of smooth Hamiltonians H_i supported in K such that:

- (1) The sequence of flows $\phi_{H_i}^t$ C^0 -converges to ϕ^t uniformly in t on every compact subset of I ,
- (2) the sequence $H_i(t, \cdot)$ C^0 -converges to a continuous function $H(t, \cdot)$ uniformly in t on every compact subset of I .

We say that H generates ϕ^t , denote $\phi^t = \phi_H^t$, and call H a *continuous Hamiltonian*. We denote by $C_{\text{Ham}}^0(M, \omega)$ (or just C_{Ham}^0) the set of all continuous functions $H: \mathbb{S}^1 \times M \rightarrow \mathbb{R}$ which, seen as functions defined on $\mathbb{R} \times M$, generate hameotopies parametrized by \mathbb{R} . A homeomorphism is called a *hameomorphism* if it is the time-1 map of some hameotopy parametrized by $[0, 1]$.

A continuous function $H \in C_{\text{Ham}}^0$ generates a unique hameotopy [21]. Conversely, Viterbo [25] and Buhovsky–Seyfaddini [3] proved that a hameotopy has a unique (up to addition of a function of time) continuous generator.

One can easily check that generators of hameotopies satisfy the same composition formulas as their smooth counterparts. Namely, if ϕ_H^t is a hameotopy, then $(\phi_H^t)^{-1}$ is a hameotopy generated by $-H(t, \phi_H^t(x))$; given another hameotopy ϕ_K^t , the isotopy $\phi_H^t \phi_K^t$ is also a hameotopy, generated by $H(t, x) + K(t, (\phi_H^t)^{-1}(x))$.

Moreover, we will repeatedly use the following simple fact: If $H \in C_{\text{Ham}}^0(V)$ for some open set V in a symplectic manifold (M, ω) and if $\theta: U \rightarrow V$ is a symplectic homeomorphism, then $H \circ \theta$ belongs to $C_{\text{Ham}}^0(U)$ and generates the hameotopy $\theta^{-1} \phi_H^t \theta$. This, in particular, holds for smooth $H: \mathbb{S}^1 \times M \rightarrow \mathbb{R}$ supported in V .

2.2. Lagrangian spectral invariants. In [24], Viterbo defined Lagrangian spectral invariants on \mathbb{R}^{2n} and cotangent bundles via generating functions. Then Oh [18] defined similar invariants via Lagrangian Floer homology in cotangent bundles which have been proven to coincide with Viterbo’s invariants by Milinković [16]. They have been adapted to the compact case by Leclercq [12] for weakly exact Lagrangians and Leclercq–Zapolsky [13] for monotone Lagrangians. However, for the type of problems which we consider

here (C^0 -convergence of Lagrangians), we can restrict ourselves to Weinstein neighborhoods and thus work only in cotangent bundles. We briefly outline below the construction of these invariants via Lagrangian Floer homology and collect their main properties in this situation. We refer to Monzner–Vichery–Zapolsky [17] which gives a very nice exposition of the theory.

Let L be a smooth compact manifold, L_0 denote the 0-section in T^*L , and λ the Liouville 1-form. To a compactly supported smooth time-dependent Hamiltonian $H \in C_c^\infty([0, 1] \times T^*L)$ is associated the action functional

$$\mathcal{A}_H: \Omega(T^*L) \rightarrow \mathbb{R}, \quad \gamma \mapsto \int_0^1 H_t(\gamma(t)) dt - \int \gamma^* \lambda$$

where $\Omega(T^*L) = \{\gamma: [0, 1] \rightarrow T^*L \mid \gamma(0) \in L_0, \gamma(1) \in L_0\}$. The critical points of \mathcal{A}_H are the chords of the Hamiltonian vector field X_H which start and end on L_0 . The spectrum of \mathcal{A}_H , denoted $\text{Spec}(\mathcal{A}_H)$, consists of the critical values of \mathcal{A}_H . It is a nowhere dense subset of \mathbb{R} which only depends on the time-1 Hamiltonian diffeomorphism ϕ_H^1 .

Following Floer's construction, for a generic choice of Hamiltonian function, $\text{crit}(\mathcal{A}_H)$ is finite and one can form a chain complex $(CF_*(H), \partial_{H,J})$ whose generators are the critical chords and whose differential counts the elements of the 0-dimensional component of moduli spaces of Floer trajectories (i.e pseudo-holomorphic curves perturbed by H) which run between the critical chords (with boundary conditions on L_0). The differential relies on the additional data of a generic enough pseudo-complex structure, J .

This complex is filtered by the values of the action, that is, for $a \in \mathbb{R}$ a regular value of \mathcal{A}_H , one can consider only chords of action less than a . Such chords generate a subcomplex of the total complex $CF_*^a(H)$ (because the action decreases along Floer trajectories). We denote by i_*^a the inclusion $CF_*^a(H) \rightarrow CF_*(H)$. By considering homotopies between pairs (H, J) and (H', J') , one can canonically identify the homology induced by the respective Floer complexes $H_*(CF(H), \partial_{H,J})$ and $H_*(CF(H'), \partial_{H',J'})$ and by considering C^2 -small enough Hamiltonian functions, one can see that the resulting object $HF_*(L_0)$ is canonically isomorphic to the singular homology of L .

Thus, one can consider spectral invariants associated to any non-zero homology class α of L , defined as the smallest action level which detects α :

$$\ell(\alpha; H) = \inf\{a \in \mathbb{R} \mid \alpha \in \text{im}(H_*(i^a))\}$$

In what follows we will only be interested in the spectral invariants associated to the class of a point and the fundamental class which will be respectively denoted by $\ell_-(H) = \ell([\text{pt}]; H)$ and $\ell_+(H) = \ell([L]; H)$.

These invariants were proven to be continuous with respect to the C^0 -norm on Hamiltonian functions so that they are defined for any (not necessarily generic) Hamiltonian. Moreover, they only depend on the time-1 map ϕ_H^1 induced by the flow of H ; hence they are well-defined on $\text{Ham}^c(T^*L, d\lambda)$.

Their main properties are collected in the following theorem, which corresponds to [17, Theorem 2.20], except for (7) which we prove below. Note that, except for (6), these properties already appear in Viterbo [24].

Theorem 13. *Let L be a smooth closed connected manifold. Let L_0 denote the 0-section of T^*L . There exist two maps $\ell_{\pm}: \text{Ham}^c(T^*L, d\lambda) \rightarrow \mathbb{R}$ with the following properties:*

- (1) *For any $\phi \in \text{Ham}^c(T^*L, d\lambda)$, $\ell_{\pm}(\phi)$ lie in $\text{Spec}(\mathcal{A}_{\phi})$.*
- (2) *$\ell_- \leq \ell_+$.*
- (3) *For any two Hamiltonian functions H and K ,*

$$\int_0^1 \min(H_t - K_t) dt \leq \ell_{\pm}(\phi_H^1) - \ell_{\pm}(\phi_K^1) \leq \int_0^1 \max(H_t - K_t) dt.$$
- (4) *For any ϕ and $\phi' \in \text{Ham}^c(T^*L, d\lambda)$, $\ell_+(\phi\phi') \leq \ell_+(\phi) + \ell_+(\phi')$.*
- (5) *For any $\phi \in \text{Ham}^c(T^*L, d\lambda)$, $\ell_{\pm}(\phi) = -\ell_{\mp}(\phi^{-1})$.*
- (6) *If $H|_{L_0} \leq c$ (respectively $H|_{L_0} \geq c$ or $H|_{L_0} = c$), then $\ell_{\pm}(\phi_H^1) \leq c$ (respectively $\ell_{\pm}(\phi_H^1) \geq c$ or $\ell_{\pm}(\phi_H^1) = c$).*
- (7) *If f is a L_0 -slow admissible Hamiltonian, then $\ell_+(\phi_f^1) = \max(f|_{L_0})$ and $\ell_-(\phi_f^1) = 0$.*
- (8) *For any ϕ and $\phi' \in \text{Ham}^c(T^*L, d\lambda)$ such that $\phi(L_0) = \phi'(L_0)$, $\ell_+(\phi) - \ell_-(\phi) = \ell_+(\phi') - \ell_-(\phi')$.*

Proof of item (7). Since f is L_0 -slow, $\text{Spec}(\phi_f^1)$ consists of critical values of f corresponding to critical points lying in L_0 . Now for all $s \in [0, 1]$, since f is autonomous, $\text{Spec}(\phi_f^s) = \text{Spec}(\phi_{sf}^1) = s \cdot \text{Spec}(\phi_f^1)$. Since in cotangent bundles spectral invariants lie in the spectrum regardless of degeneracy of f , by continuity of ℓ_{\pm} there exist $p_{\pm} \in \text{crit}(f) \cap L_0$ such that $\ell_{\pm}(\phi_f^s) = s \cdot f(p_{\pm})$. Now for small times s , sf is C^2 -small and thus $\ell_+(\phi_{sf}^1)$ (respectively $\ell_-(\phi_{sf}^1)$) is the maximum (respectively the minimum) of sf so that $f(p_+) = \max(f)$ (respectively $f(p_-) = \min(f)$). \square

In view of Property (8), Viterbo (followed by Oh) derived an invariant γ of Lagrangian Hamiltonian isotopic to the 0-section, defined as follows.

Definition 14. *For a Hamiltonian diffeomorphism $\phi \in \text{Ham}^c(T^*L, d\lambda)$ we set $\gamma(\phi) = \ell_+(\phi) - \ell_-(\phi)$ and for a Lagrangian L Hamiltonian isotopic to the 0-section, we set $\gamma(L, L_0) = \gamma(\phi)$ for any $\phi \in \text{Ham}^c(T^*L, d\lambda)$ such that $\phi(L_0) = L$.*

From the properties of spectral invariants, it is immediate that for all ϕ and $\psi \in \text{Ham}^c(T^*L, d\lambda)$, $0 \leq \gamma(\phi\psi) \leq \gamma(\phi) + \gamma(\psi)$, and that $\gamma(\phi) = \gamma(\phi^{-1})$. Moreover, $\gamma(L, L_0) = 0$ implies $L = L_0$ as proven in [24].

The main property of γ which will be used in what follows is the fact that

$$(1) \quad \gamma(\phi_H^1(L_0), L_0) = \gamma(\phi_H^1) \leq \max_{t \in [0, 1]} (\text{osc}(H_t|_{L_0}))$$

where $\text{osc}(H_t|_{L_0}) = \max_{L_0}(H_t) - \min_{L_0}(H_t)$. This inequality can be directly derived from Property (6) of Theorem 13. Note that this yields

$$\forall t_0 \in \mathbb{R}, \quad \gamma(\phi_H^{t_0}(L_0), L_0) \leq t_0 \cdot \max_{t \in [0, t_0]} (\text{osc}(H_t|_{L_0})).$$

3. ENERGY-CAPACITY INEQUALITIES

3.1. The energy-capacity inequality for Hamiltonians constant on open sets. We prove Lemma 8 by mimicking the proof of the corresponding inequality in [10] (here for Lagrangians, in the easier world of aspherical objects). Then we prove a corollary which will be used in the proof of the main result.

Proof of Lemma 8. First, assume that $\ell_-(\phi_H^1) \leq 0$. Then for any admissible L_0 -slow function with support in U_+ , $f \in \mathcal{H}(U_+)$, we define the 1-parameter family of Hamiltonians $H_s(t, x) = H(t, x) - sf(x)$ with $s \in [0, 1]$. Since H is constant on U_+ , H_s generates $\phi_{H_s}^1 = \phi_H^1 \phi_f^{-s}$. By triangle inequality and duality (i.e Properties (4) and (5)) of spectral invariants, we get

$$(2) \quad \ell_+(\phi_f^s) \leq \ell_+(\phi_{H_s}^{-1}) + \ell_+(\phi_H^1) = \ell_+(\phi_H^1) - \ell_-(\phi_{H_s}^1).$$

Then notice that

$$\text{Spec}(\mathcal{A}_{H_s}) = \text{Spec}(\mathcal{A}_H) \cup \{C_+ - sf(p) \mid p \in \text{crit}(f) \cap U_+\}.$$

Since for all $p \in \text{crit}(f) \cap U_+$, $sf(p) \leq \max(f) \leq c_{\text{LR}}(U_+; L_0) < C_+$, the non-positive spectrum of \mathcal{A}_{H_s} , $\text{Spec}_-(\mathcal{A}_{H_s}) = \text{Spec}(\mathcal{A}_{H_s}) \cap \mathbb{R}_-$, does not depend on s and coincides with $\text{Spec}_-(\mathcal{A}_H)$ which is totally discontinuous. Since the map $s \mapsto \ell_-(\phi_{H_s}^1)$ is continuous and maps 0 to $\text{Spec}_-(\mathcal{A}_H)$, it has to be constant. Thus $\ell_-(\phi_{H_1}^1) = \ell_-(\phi_H^1)$ and, from (2), Property (7) of spectral invariants immediately leads to

$$\max(f) = \ell_+(\phi_f^1) \leq \ell_+(\phi_H^1) - \ell_-(\phi_H^1) = \gamma(\phi_H^1(L_0), L_0).$$

Since this holds for any L_0 -slow function in $\mathcal{H}(U_+)$, we get $\gamma(\phi_H^1(L_0), L_0) \geq c_{\text{LR}}(U_+; L_0)$.

Now, assume that $\ell_-(\phi_H^1) \geq 0$ and consider $\bar{H}(t, x) = -H(t, \phi_H^t(x))$. By assumption, ϕ_H^t is the identity on U_- and $\bar{H}|_{U_-} = -H|_{U_-} = C_-$. Since \bar{H} generates $\phi_{\bar{H}}^1 = (\phi_H^1)^{-1}$,

$$\ell_-(\phi_{\bar{H}}^1) = -\ell_+(\phi_H^1) \leq -\ell_-(\phi_H^1) \leq 0.$$

Then the first case gives that $\gamma(\phi_{\bar{H}}^1) \geq c_{\text{LR}}(U_-; L_0)$ which concludes the proof since $\gamma(\phi_{\bar{H}}^1) = \gamma(\phi_H^1) = \gamma(\phi_H^1(L_0), L_0)$. \square

From this (and as in [10]) we infer the same result but for Hamiltonian functions which are allowed to have controlled oscillations on U_{\pm} .

Corollary 15. *Let L be a smooth closed manifold. Let U_- and U_+ be open subsets of T^*L such that $U_{\pm} \cap L_0 \neq \emptyset$. Assume that $c_{\text{LR}}(U_-; L_0) = c_{\text{LR}}(U_+; L_0)$. Let H be a Hamiltonian so that for all t in $[0, 1]$*

- (1) $\inf(H_t|_{U_+}) > c_{\text{LR}}(U_{\pm}; L_0)$, and $\sup(H_t|_{U_-}) < -c_{\text{LR}}(U_{\pm}; L_0)$,
 (2) $\text{osc}(H_t|_{U_{\pm}}) < \frac{1}{3}c_{\text{LR}}(U_{\pm}; L_0)$.

Then, $\gamma(\phi_H^1(L_0), L_0) \geq \frac{1}{3}c_{\text{LR}}(U_{\pm}; L_0)$.

Proof. Fix $\varepsilon > 0$. We choose disjoint open subsets V_{\pm} such that $U_{\pm} \Subset V_{\pm}$ (with \Subset denoting compact containment) and $\text{osc}_{V_{\pm}}(H) < \text{osc}_{U_{\pm}}(H) + \varepsilon$. We also choose cut-off functions ρ_{\pm} with support in V_{\pm} , such that $0 \leq \rho_{\pm} \leq 1$ and $\rho_{\pm}|_{U_{\pm}} = 1$. Then we define

$$h = H - \rho_+(H - C_+) - \rho_-(H + C_-)$$

with $C_+ = \inf(H|_{[0,1] \times U_+})$ and $C_- = -\sup(H|_{[0,1] \times U_-})$. By triangle inequality, we get $\gamma(\phi_H^1) \geq \gamma(\phi_h^1) - \gamma((\phi_H^1)^{-1}\phi_h^1)$ and we now bound the right-hand side terms.

First, notice that h satisfies the requirements of Lemma 8: $h|_{U_{\pm}} = \pm C_{\pm}$ with by assumption

$$C_+ > c_{\text{LR}}(U_{\pm}; L_0), \text{ and } C_- > c_{\text{LR}}(U_{\pm}; L_0).$$

Thus we immediately get that $\gamma(\phi_h^1) \geq c_{\text{LR}}(U_{\pm}; L_0)$. Now by Inequality (1)²,

$$\gamma((\phi_H^1)^{-1}\phi_h^1) \leq \text{osc}_{L_0}((H - h) \circ (\phi_H)^{-1}) \leq \text{osc}_{V_- \cup V_+}(H - h)$$

so that

$$\begin{aligned} \gamma((\phi_H^1)^{-1}\phi_h^1) &\leq \text{osc}_{V_+}(\rho_+(H - C_+)) + \text{osc}_{V_-}(\rho_-(H + C_-)) \\ &\leq \text{osc}_{V_+}(H) + \text{osc}_{V_-}(H) \leq \text{osc}_{U_+}(H) + \text{osc}_{U_-}(H) + 2\varepsilon \\ &\leq \frac{2}{3}c_{\text{LR}}(U_{\pm}; L_0) + 2\varepsilon. \end{aligned}$$

So $\gamma(\phi_H^1(L_0), L_0) \geq \frac{1}{3}c_{\text{LR}}(U_{\pm}; L_0) - 2\varepsilon$ for any $\varepsilon > 0$. \square

3.2. The energy-capacity inequality for Lagrangians displaced from an open set. We give a proof of Lemma 9 from Lisi–Rieser [15], for the reader's convenience. The method of proof is now classical and goes back to Viterbo [24] (see also Usher's proof of the analogous result [23] for compact manifolds, itself heavily influenced by Frauenfelder–Ginzburg–Schlenk [6]). Recall that in cotangent bundles the spectral invariants only depend on the endpoint of Hamiltonian isotopies; this drastically simplifies the proof.

Proof of Lemma 9. Assume that $\phi_H^1(L_0) \cap U = \emptyset$. Choose a L_0 -slow function f in $\mathcal{H}(U)$. For $s \in [0, 1]$, consider the Hamiltonian diffeomorphism $\phi_{sf}^1 \phi_H^1$. It is the end of the isotopy defined as the concatenation

$$\theta^t = \begin{cases} \phi_H^{2t}, & t \in [0, 1/2], \\ \phi_{sf}^{2t-1} \phi_H^1, & t \in [1/2, 1]. \end{cases}$$

An Hamiltonian chord of θ is a path $t \mapsto \gamma(t)$ such that $\gamma(0) \in L_0$, $\gamma(1) \in L_0$ and for all t , $\gamma(t) = \theta^t(\gamma(0))$. In particular, for such a chord, $\phi_{sf}^1 \phi_H^1(\gamma(0)) \in L_0$. However, since by assumption $\phi_H^1(L_0) \cap \text{supp}(sf) = \emptyset$, necessarily

²In the next few lines we loosely denote $\max_{t \in [0,1]} \text{osc}(f_t|_U)$ by $\text{osc}_U(f)$ for readability.

$\phi_H^1(\gamma(0))$ is not in the support of f and $\gamma(t) = \phi_H^{2t}(\gamma(0))$ for all $t \leq 1/2$ and remains constant $\gamma(t) = \phi_H^1(\gamma(0))$ for $t \geq 1/2$.

This means that for all s , the set of Hamiltonian chords remains constant and so does $\text{Spec}(\phi_{sf}^1 \phi_H^1)$. Since this set is nowhere dense and ℓ_+ is continuous (and takes its values in the action spectrum), the function $s \mapsto \ell_+(\phi_{sf}^1 \phi_H^1)$ is constant so that $\ell_+(\phi_H^1) = \ell_+(\phi_f^1 \phi_H^1)$. Thus,

$$\ell_+(\phi_f^1) \leq \ell_+(\phi_f^1 \phi_H^1) + \ell_+(\phi_H^1)^{-1} = \ell_+(\phi_H^1) - \ell_-(\phi_H^1) = \gamma(\phi_H^1)$$

by Properties (4) and (5) of spectral invariants. Since this holds for any L_0 -slow function $f \in \mathcal{H}(U)$ and since for such a function $\max(f) = \ell_+(\phi_f^1)$ by Property (7) of spectral invariants, the result follows. \square

3.3. An upper bound for the spectral distance. In this section we prove Lemma 10 which establishes Viterbo's conjecture in a special case. The fact that Viterbo's conjecture holds under the additional assumptions of Lemma 10 seems to be well known to experts; we provide a proof here for the sake of completeness.

Proof of Lemma 10. Note that since $\phi_H^t(L_0) \subset T_r^*L$ for all $t \in [0, 1]$, modifying H outside of T_r^*L leaves $\phi_H^1(L_0)$, and hence $\gamma(\phi_H^1(L_0), L_0)$, unchanged. Therefore, by cutting H off outside of T_r^*L and replacing r with $2r$ we may assume that H is supported inside $T_r^*L \setminus V$.

Pick $f: L \rightarrow \mathbb{R}$ to be a Morse function on L whose critical points are all contained in the open set \mathcal{V} . Because f has no critical points inside $L \setminus \mathcal{V}$, we may assume, by rescaling, that

$$(3) \quad \|df|_{L \setminus \mathcal{V}}\|_g \geq 1.$$

Let $\beta: T^*L \rightarrow \mathbb{R}$ denote a non-negative cutoff function such that $\beta = 1$ on T_R^*L , where R is picked so that $R \gg r$. Let $F = \beta\pi^*f: T^*L \rightarrow \mathbb{R}$. By picking R to be sufficiently large, we can ensure that, for $t \in [0, 1]$ and (q, p) in a neighborhood of T_r^*L , the Hamiltonian flow of F is given by the formula

$$\phi_F^t(q, p) = (q, p + tdf(q)).$$

This, combined with (3), implies that $\phi_F^t(q, 0) \notin T_r^*L$ for any $t > r$ and any point $q \in L \setminus \mathcal{V}$. Hence, we see that $\phi_F^t(L_0)$ is outside the support of H for any $t > r$. Therefore, $\phi_H^1 \phi_F^{2r}(L_0) = \phi_F^{2r}(L_0)$, and so

$$\gamma(\phi_H^1 \phi_F^{2r}(L_0), L_0) = \gamma(\phi_F^{2r}(L_0), L_0) \leq 2r \text{osc}(F) = 2r \text{osc}(f).$$

The Hamiltonian diffeomorphism $\phi_H^1 \phi_F^{2r}$ is the time-1 map of the flow of $G(t, x) = H(t, x) + 2rF(\phi_H^t(x))$. Using Property (3) from Theorem 13, we obtain that

$$\min(2rF) \leq \ell_{\pm}(\phi_H^1 \phi_F^{2r}) - \ell_{\pm}(\phi_H^1) \leq \max(2rF),$$

from which we conclude that $|\gamma(\phi_H^1 \phi_F^{2r}(L_0), L_0) - \gamma(\phi_H^1(L_0), L_0)| \leq 2r \text{osc}(F) = 2r \text{osc}(f)$. The result follows with $C = 4 \text{osc}(f)$. \square

4. LOCALIZED RESULTS FOR C^0 -LAGRANGIANS

Recall from Definition 3 that L is called a C^0 -Lagrangian if around each point $p \in L$ there exists a C^0 -Lagrangian chart, that is, a symplectic homeomorphism $\theta: U \rightarrow V \subset \mathbb{R}^{2n}$, with U open neighborhood of p , so that $\theta(L \cap U) = \mathcal{L}_0 \cap V$ where

$$\mathcal{L}_0 = \{(x_1, \dots, x_n, y_1, \dots, y_n) \mid (y_1, \dots, y_n) = (0, \dots, 0)\}.$$

The main goal of this section is to establish a suitable localized version of Theorem 6 for C^0 -Lagrangians; since we seek localized statements we do not assume that the C^0 -Lagrangians in question are necessarily closed. Not surprisingly, in this new setting Theorem 6 does not hold as stated. The localized results of this section, which have more complicated statements and proofs, are more powerful and they constitute the main technical steps towards proving Theorems 1 and 6.

We prove the analog of the direct implication of Theorem 6 in Section 4.1. The analog of the converse implication is proven in Section 4.2. Since L is a C^0 -Lagrangian, its C^0 -characteristic foliation has a single leaf, which is L itself, and thus the results of this section make no mention of C^0 -characteristic foliations.

4.1. C^0 -Hamiltonians constant on a Lagrangian preserve it. In this subsection, we show that if the restriction of $H \in C_{\text{Ham}}^0$ to a (not necessarily closed) C^0 -Lagrangian L is constant, then the associated hamotopy ϕ_H preserves L , locally. More precisely,

Proposition 16. *Let $L \subset M$ denote a C^0 -Lagrangian (not necessarily closed) and $H \in C_{\text{Ham}}^0$ with associated hamotopy ϕ_H . If $H|_L = c(t)$, a function of time, then for any point $p \in L$ there exists $\varepsilon > 0$ such that $\phi_H^t(p) \in L$ for all $t \in [0, \varepsilon]$.*

Our proof of the above proposition will use the following lemma on the local structure of C^0 -Lagrangians.

Lemma 17. *Let $L \subset M$ denote a C^0 -Lagrangian. Around each point $p \in L$ there exists a C^0 -Lagrangian $L_p \subset L$ and a C^0 -Lagrangian chart (U, θ) such that L_p is compactly contained in U , and $\theta(L_p) \subset T \subset \theta(U)$, where T denotes a Lagrangian torus in \mathbb{R}^{2n} .*

Proof. Let (U, θ) denote a C^0 -Lagrangian chart around p such that $\theta(U) = \{(x_i, y_i) \mid -a < x_i, y_i < a\}$ and $\theta(L \cap U) = \{(x_i, y_i) \mid -a < x_i < a, y_i = 0\}$. Let $L_p = \theta^{-1}(\{(x_i, y_i) \mid -\frac{a}{2} \leq x_i \leq \frac{a}{2}, y_i = 0\})$. Then, for all k , the projection of $\theta(L_p)$ onto the x_k coordinate is the interval $[-\frac{a}{2}, \frac{a}{2}]$ which can be completed to a smooth embedded loop, say T_k , in the $\langle x_k, y_k \rangle$ plane. Furthermore, we can ensure that T_k is contained in the projection of $\theta(U)$ onto the $\langle x_k, y_k \rangle$ plane. We set $T = T_1 \times \dots \times T_n$. \square

Proof of Proposition 16. By replacing H by $H - c(t)$ we can suppose that $H|_L = 0$. Apply Lemma 17 to obtain L_p as described in the lemma and

note that by replacing L with L_p we may make the following simplifying assumption:

L is a C^0 -Lagrangian covered by a C^0 -Lagrangian chart (U, θ) such that $\theta(L) \subset T \subset \theta(U)$ where T is a (standard) Lagrangian torus in \mathbb{R}^{2n} .

We will now prove the proposition under this simplifying assumption. Let $H_i: [0, 1] \times M \rightarrow \mathbb{R}$ denote a sequence of smooth Hamiltonians such that H_i converges uniformly to H and ϕ_{H_i} converges to ϕ_H in C^0 -topology. Take $W \Subset V$ to be proper open subsets of U such that

$$(4) \quad p \in W \cap L \quad \text{and} \quad \overline{V} \cap (\theta^{-1}(T) \setminus L) = \emptyset.$$

Recall that the symbol \Subset denotes compact containment and \overline{V} denotes the closure of V . Denote by $\beta: M \rightarrow \mathbb{R}$ a cutoff function such that β is supported in U , $\beta|_V = 1$, and $\beta|_{\theta^{-1}(T) \setminus L} = 0$. By shrinking V , if needed, we may assume that θ maps the support of β into a Weinstein neighborhood of T .

Let $G_i = \beta H_i$ and $G = \beta H$. Observe that G_i converges uniformly to G , $G|_{\theta^{-1}(T)} = 0$. We pick $\varepsilon > 0$ such that

$$(5) \quad \forall t \in [0, \varepsilon], \quad \phi_H^t(W) \subset V.$$

For i large enough $\phi_{H_i}^t(W) \subset V$ for all $t \in [0, \varepsilon]$. Since, $G_i|_V = H_i|_V$ we conclude that, for large i ,

$$(6) \quad \forall (t, x) \in [0, \varepsilon] \times W, \quad \phi_{G_i}^t = \phi_{H_i}^t.$$

For a contradiction, suppose that $\phi_H^{t_0}(p)$ is not contained in L for some $t_0 \in [0, \varepsilon]$. From (4) and (5) we conclude that $\phi_H^{t_0}(p) \notin \theta^{-1}(T)$. Hence, we can find a small ball $B \subset W$ around p which intersects $\theta^{-1}(T)$ non-trivially and such that $\overline{\phi_H^{t_0}(B)} \cap \theta^{-1}(T) = \emptyset$. Hence, for i large enough, we have $\phi_{H_i}^{t_0}(B) \cap \theta^{-1}(T) = \emptyset$. From (6) we get that $\phi_{G_i}^{t_0}(B) \cap \theta^{-1}(T) = \emptyset$.

Now, let $F_i = G_i \circ \theta^{-1}$, and $F = G \circ \theta^{-1}$. The sequence F_i converges uniformly to F and $F|_T = 0$. Furthermore, $\phi_{F_i}^{t_0} = \theta \phi_{G_i}^{t_0} \theta^{-1}$. Hence, it follows that $\phi_{F_i}^{t_0}(\theta(B)) \cap T = \emptyset$. This is, of course, equivalent to

$$(7) \quad (\phi_{F_i}^{t_0})^{-1}(T) \cap \theta(B) = \emptyset.$$

We picked β such that the Hamiltonians F_i all have support in a Weinstein neighborhood of T . Hence, we can pass to T^*T , and work with the Lagrangian spectral invariants of the 0-section T_0 , associated to these Hamiltonians; see Section 2.2. From (7) and Lemma 9, we get that

$$\gamma(\phi_{F_i}^{t_0}(T_0), T_0) = \gamma((\phi_{F_i}^{t_0})^{-1}(T_0), T_0) \geq c_{\text{LR}}(\theta(B); T_0) > 0.$$

Inequality (1) from Section 2.2 implies that

$$t_0 \cdot \max_{t \in [0, t_0]} (\text{osc}(F_i(t, \cdot)|_{T_0})) \geq c_{\text{LR}}(\theta(B); T_0).$$

Since F_i converges uniformly to F , the same inequality must hold for F but this contradicts the fact that $F|_T = 0$. \square

4.2. C^0 -Hamiltonians preserving a Lagrangian are constant on it.

In this subsection, we show that if $H \in C_{\text{Ham}}^0$ generates an hameotopy ϕ_H which (locally) preserves a C^0 -Lagrangian L then, the restriction of H to L is (locally) constant. More precisely,

Proposition 18. *Let $L \subset M$ denote a C^0 -Lagrangian, \mathcal{U} an open subset of L , and $H \in C_{\text{Ham}}^0$ with associated hameotopy ϕ_H . Suppose that $\phi_H^t(\mathcal{U}) \subset L$ for all $t \in [0, 1]$ and let \mathcal{V} denote the interior of $\cap_{t \in [0, 1]} \phi_H^t(\mathcal{U})$. Then, $H(t, \cdot)|_{\mathcal{V}}$ is a locally constant function for each $t \in [0, 1]$.*

We were recently informed by Y.-G. Oh that it is possible to extract the above proposition from [19, Theorem 4.9]; the techniques of [19] are different than ours.

Our proof of the proposition uses the following consequence of Corollary 15. This result can be viewed as a Lagrangian analog of the uniqueness of generators Theorem [10, Theorem 2]. The argument presented here is similar to the proof of the mentioned uniqueness theorem.

Proposition 19. *Let L be a smooth closed manifold and $\{H_k\}_k$ a sequence of smooth, uniformly compactly supported—that is, there exists a compact K such that $\cup_k \text{supp}(H_k) \subset K$ —, Hamiltonian functions on T^*L so that*

- (1) *for all $t \in [0, 1]$, $\gamma(\phi_{H_k}^t(L_0), L_0)$ converges to 0, and*
- (2) *$\{H_k\}_k$ uniformly converges to a continuous function H .*

Then, H restricted to L_0 is a function of time.

Proof. If $H|_{[0, 1] \times L_0}$ is not a function of time, there exist t_0 and $x_+, x_- \in L_0$ such that $H_{t_0}(x_+) > H_{t_0}(x_-)$. Up to a shift (and cutoff far from K), we can assume that $H_{t_0}(x_+) = -H_{t_0}(x_-) = \Delta > 0$.

Now, notice that there exist $\delta_0 \in (0, 1]$ and $r_0 > 0$ such that $J^{\delta_0} = [t_0, t_0 + \delta_0] \subset [0, 1]$ and that there exist symplectically embedded balls, centered at x_{\pm} , $B_{\pm}^{r_0} = \iota_{\pm}(B_{\mathbb{C}^n}(0, r_0))$, with real part mapped to L_0 , which are disjoint and such that

$$(8) \quad \left(\frac{5}{4} \frac{1}{M_+^{\delta_0, r_0}}, \frac{4}{3} \frac{1}{M_+^{\delta_0, r_0}} \right) \cap \left(\frac{5}{4} \frac{1}{M_-^{\delta_0, r_0}}, \frac{4}{3} \frac{1}{M_-^{\delta_0, r_0}} \right) \neq \emptyset$$

with $M_+^{\delta_0, r_0} = \sup_{J^{\delta_0} \times B_+^{r_0}}(H)$ and $M_-^{\delta_0, r_0} = -\inf_{J^{\delta_0} \times B_-^{r_0}}(H)$. To ensure that (8) holds, choose r_0 and δ_0 small enough such that

$$\sup_{J^{\delta_0} \times B_+^{r_0}}(H) \leq H_{t_0}(x_+) + \eta \quad \text{and} \quad \inf_{J^{\delta_0} \times B_-^{r_0}}(H) \geq H_{t_0}(x_-) - \eta$$

with $\eta = \frac{\Delta}{16}$. Then $\Delta \leq M_{\pm}^{\delta_0, r_0} \leq \Delta + \eta$, so that $|M_+^{\delta_0, r_0} - M_-^{\delta_0, r_0}| \leq \eta = \frac{\Delta}{16}$ and (8) follows.

For $r \leq r_0$, we denote by $B_{\pm}^r \subset B_{\pm}^{r_0}$ the embeddings of the smaller balls, $\iota_{\pm}(B_{\mathbb{C}^n}(0, r))$, and by $J^{\delta} = [t_0, t_0 + \delta]$ for $\delta \leq \delta_0$. Note that the previous properties continue to hold for any choice of $r \leq r_0$ and $\delta \leq \delta_0$.

Now, assume that $\delta \leq \delta_0$ and $r \leq r_0$ are small enough such that

$$\inf_{J^\delta \times B_+^r} (H) > \frac{4}{5} M_+^{\delta, r} \quad \text{and} \quad - \sup_{J^\delta \times B_-^r} (H) > \frac{4}{5} M_-^{\delta, r}$$

(this can be achieved since these inequalities hold for $\delta = 0$ and $r = 0$ and H is continuous). We fix such δ .

Notice that $c_{\text{LR}}(B_+^r; L_0) = c_{\text{LR}}(B_-^r; L_0) = \frac{\pi r^2}{2}$ (we denote this common value $c_{\text{LR}}(B_\pm^r; L_0)$) and that we can choose r small enough such that $c_{\text{LR}}(B_\pm^r; L_0) < \delta \frac{3}{4} \Delta$ which yields $c_{\text{LR}}(B_\pm^r; L_0) < \delta \frac{3}{4} M_\pm^{\delta, r}$. Now that r is also fixed, we remove δ, r from the notation and we choose σ such that

$$\frac{\delta \sigma}{c_{\text{LR}}(B_\pm; L_0)} \in \left(\frac{5}{4} \frac{1}{M_+}, \frac{4}{3} \frac{1}{M_+} \right) \cap \left(\frac{5}{4} \frac{1}{M_-}, \frac{4}{3} \frac{1}{M_-} \right).$$

Notice that $\sigma < 1$ and define F_k by $F_k(t, x) = \sigma \delta H_k(t_0 + \sigma \delta t, x)$ and F accordingly. By our choices of constants,

$$\inf_{B_+} (F) > c_{\text{LR}}(B_\pm; L_0) \quad \text{and} \quad \sup_{B_+} (F) < \frac{4}{3} c_{\text{LR}}(B_\pm; L_0)$$

which, in particular, implies that $\text{osc}_{B_+} (F) < \frac{1}{3} c_{\text{LR}}(B_\pm; L_0)$. Equivalently, on B_- we obtain

$$- \sup_{B_-} (F) > c_{\text{LR}}(B_\pm; L_0) \quad \text{and} \quad \text{osc}_{B_-} (F) < \frac{1}{3} c_{\text{LR}}(B_\pm; L_0).$$

Thus, by Corollary 15, $\gamma(\phi_F^1(L_0), L_0) \geq \frac{1}{3} c_{\text{LR}}(B_\pm; L_0)$. Since $\{F_k\}$ C^0 -converges to F , by Property (3) of spectral invariants, $\gamma(\phi_{F_k}^1(L_0), L_0)$ is uniformly bounded away from 0 (say by $\frac{1}{6} c_{\text{LR}}(B_\pm; L_0)$ for k big enough). However, $\phi_{F_k}^1 = \phi_{H_k}^{t_0 + \sigma \delta} (\phi_{H_k}^{t_0})^{-1}$ so that, by Properties (4) and (5) of spectral invariants,

$$\begin{aligned} \gamma(\phi_{F_k}^1(L_0), L_0) &= \gamma(\phi_{F_k}^1) \leq \gamma(\phi_{H_k}^{t_0 + \sigma \delta}) + \gamma((\phi_{H_k}^{t_0})^{-1}) = \gamma(\phi_{H_k}^{t_0 + \sigma \delta}) + \gamma(\phi_{H_k}^{t_0}) \\ &\leq \gamma(\phi_{H_k}^{t_0 + \sigma \delta}(L_0), L_0) + \gamma(\phi_{H_k}^{t_0}(L_0), L_0) \end{aligned}$$

which goes to 0 when k goes to infinity because of Assumption (1) and we get a contradiction. \square

Proof of Proposition 18. Pick a point $p \in \mathcal{V}$ and let (U, θ) denote a C^0 -Lagrangian chart around p such that $\theta(L \cap U) = \{(x_i, y_i) \mid -a < x_i < a, y_i = 0\}$. Furthermore, by shrinking U if needed we may assume that $L \cap U \subset \mathcal{V}$. We will show $H(t, \cdot)$ is constant on $L \cap U$. For a contradiction, suppose that there exists $q \in L \cap U$ and $t_0 \in [0, 1]$ such that $H(t_0, p) \neq H(t_0, q)$.

First, note that, up to time reparametrization, we may assume that $H(0, p) \neq H(0, q)$. Indeed, replace H with $\tilde{H}(t, x) = aH(t_0 + at, x)$ where $a = 1 - t_0$. The time- t flow of \tilde{H} is given by the expression: $\phi_{\tilde{H}}^{t_0 + at} (\phi_{\tilde{H}}^{t_0})^{-1}$. Let $\tilde{\mathcal{U}} = \phi_{\tilde{H}}^{t_0}(\mathcal{U})$. Then, $L \cap U$ is contained in the interior of $\cap_{t \in [0, 1]} \phi_{\tilde{H}}^t(\tilde{\mathcal{U}})$ and $\tilde{H}(0, p) \neq \tilde{H}(0, q)$.

Let B denote an open neighborhood of p, q which is compactly contained in U . Pick a symplectic homeomorphism ψ supported in B such that ψ preserves L and $\psi(p) = q$. Such symplectic homeomorphism can be constructed as follows: Because $\theta(L \cap U) = \{(x_i, y_i) \mid -a < x_i < a, y_i = 0\}$, we can find a symplectomorphism ϕ of \mathbb{R}^{2n} such that ϕ is supported in $\theta(B)$, $\phi(\theta(p)) = \theta(q)$, and ϕ preserves $\{(x_i, y_i) \mid y_i = 0\}$. In particular, ϕ preserves $\theta(L \cap U)$. Let $\psi = \theta^{-1}\phi\theta$.

Next, we pick $\varepsilon > 0$ such that $\phi_H^t(B), (\phi_H^t)^{-1}(B) \Subset U$ for all $t \in [0, \varepsilon]$. By a reparametrization in time, where H is replaced with $\tilde{H}(t, x) = \varepsilon H(\varepsilon t, x)$, we may assume that $\phi_H^t(B), (\phi_H^t)^{-1}(B) \Subset U$ for all $t \in [0, 1]$.

Consider the continuous Hamiltonian $G = (H \circ \psi - H) \circ \phi_H$. It is supported in $\cup_{t \in [0, 1]} (\phi_H^t)^{-1}(B) \subset U$, and moreover, the flow of G is $(\phi_H^t)^{-1}\psi^{-1}\phi_H^t\psi$. We will now prove that this flow preserves L globally. Note that the flow is supported in U and pick $x \in L \cap U$. Since $\phi_H^t(L \cap U) \subset L$, and ψ, ψ^{-1} both preserve L , we see that $\phi_H^t\psi(x) \in L$. First, suppose that $\phi_H^t\psi(x) \notin B$. Then, $\phi_H^t\psi(x)$ is outside the support of ψ^{-1} and so $(\phi_H^t)^{-1}\psi^{-1}\phi_H^t\psi(x) = \psi(x)$ which is in L . Next, suppose that $\phi_H^t\psi(x) \in B \cap L$. Then, $\psi^{-1}\phi_H^t\psi(x) \in B \cap L$, and so it suffices to check that $(\phi_H^t)^{-1}(B \cap L) \subset L$: this is because $B \cap L \subset \mathcal{V} \subset \cap_{t \in [0, 1]} \phi_H^t(\mathcal{U})$, and hence, $(\phi_H^t)^{-1}(B \cap L) \subset \mathcal{U} \subset L$. We have proven that the flow of G preserves L globally.

Note that $G|_{L \cap U}$ is not a function of time only: $G(0, p) = H(0, q) - H(0, p) \neq 0$ and $G = 0$ near the boundary of U . Hence, we have obtained a C^0 -Hamiltonian G , supported in U , such that its flow ϕ_G^t preserves L globally, but $G(0, \cdot)$ is not constant on $L \cap U$. Because $G \in C_{\text{Ham}}^0$ there exist smooth Hamiltonians G_i such that $\{G_i\}$ converges uniformly to G and $\{\phi_{G_i}\}$ converges to ϕ_G . Furthermore, we can ensure that all G_i s are supported in U . This can be achieved by picking a corresponding sequence of smooth Hamiltonians H_i for H and defining $G_i = (H_i \circ \psi - H_i) \circ \phi_{H_i}$. For large i , G_i is supported in U .

Now, let $F = G \circ \theta^{-1}$ and $F_i = G_i \circ \theta^{-1}$; we extend these functions to all of \mathbb{R}^{2n} by setting them to vanish outside of $\theta(U)$. Recall that $\theta(L \cap U) = \{(x_i, y_i) \mid -a < x_i < a, y_i = 0\}$; hence we can find a Lagrangian torus T in \mathbb{R}^{2n} such that $T \cap \theta(U) = \theta(L \cap U)$. The Hamiltonian F is supported in $\theta(U)$ and it generates the hamotopy $\theta\phi_G^t\theta^{-1}$, which preserves $\theta(L \cap U)$, and hence it preserves the Lagrangian torus T . But F is not constant on T : $F(0, \theta(p)) \neq 0$, but $F(t, x) = 0$ for all $x \notin \theta(U)$. By shrinking the set U we may assume that $\theta(U)$ is contained in a Weinstein neighborhood of T . Hence, we can pass to T^*T and work with Lagrangian spectral invariants of the 0-section T_0 associated to the Hamiltonians F_i . Furthermore, the Hamiltonians F_i are all supported in $\theta(U)$ and hence we can apply Lemma 10 and conclude that, for any $r > 0$, $\gamma(\phi_{F_i}^1(T_0), T_0) \leq Cr$, i.e. $\gamma(\phi_{F_i}^1(T_0), T_0) \rightarrow 0$. Of course, by the same reasoning we obtain that $\gamma(\phi_{F_i}^t(T_0), T_0) \rightarrow 0$ for all $t \in [0, 1]$. Then, Proposition 19 implies that $F|_T = c(t)$, which contradicts the fact that $F|_T$ is not a function of time only. \square

5. C^0 -COISOTROPIC SUBMANIFOLDS AND THEIR CHARACTERISTIC FOLIATIONS

This section is devoted to the proofs of Theorems 1 and 6. We begin by proving a local version of the direct implication of Theorem 6 (Lemma 21 below) from which we deduce Theorem 1. We then prove Corollaries 4, 5 and finally Theorem 6.

Before stating the lemma, recall (Definition 3) that given a C^0 -submanifold C , a C^0 -coisotropic chart around a point $p \in C$ is a pair (θ, U) where U is an open neighborhood of p and $\theta: U \rightarrow V \subset \mathbb{R}^{2n}$ is a symplectic homeomorphism which maps C to the standard coisotropic linear subspace

$$\mathcal{C}_0 = \{(x_1, \dots, x_n, y_1, \dots, y_n) \mid (y_{n-k+1}, \dots, y_n) = (0, \dots, 0)\}.$$

Lemma 20. *Let (M, ω) be a symplectic manifold and C a C^0 -coisotropic submanifold of M . Let $H \in C_{\text{Ham}}^0(M, \omega)$ with induced hamotopy ϕ_H . Assume that the restriction of H to C only depends on time. Let $p \in C$ and $\theta: U \rightarrow V$ be a C^0 -coisotropic chart around p such that $\theta(p) = 0$. Then there exists $\varepsilon > 0$ such that for any $t \in [0, \varepsilon]$, $\phi_H^t(p) \in \theta^{-1}(\mathcal{F}_0(0))$, where $\mathcal{F}_0(0)$ is the characteristic leaf of \mathcal{C}_0 through 0.*

Before going into the details of the proof of Lemma 20, it is interesting to make the following observation. The lemma holds for coisotropic submanifolds of arbitrary codimension but its proof will follow from the particular case of Lagrangians. As mentioned in the introduction, this is not surprising in view of Weinstein's creed: "Everything is a Lagrangian submanifold!" [27].

Proof. For $i \in \{1, \dots, n-k\}$ consider the Lagrangian subspaces

$$\Lambda_i = \{(x_1, \dots, x_n, y_1, \dots, y_n) \mid x_i = 0 \text{ and } \forall j \neq i, y_j = 0\}.$$

Clearly, for all $i \in \{1, \dots, n-k\}$, $\Lambda_i \subset \mathcal{C}_0$ and

$$\mathcal{F}_0(0) = \bigcap_{i=1}^{n-k} \Lambda_i.$$

Let H be as in the statement of Lemma 20. Then for any i , the restriction of H to the C^0 -lagrangians $L_i = \theta^{-1}(\Lambda_i \cap V)$ is a function of time since L_i is included in C . Thus by Proposition 16 there exists $\varepsilon_i > 0$ such that for all $t \in [0, \varepsilon_i]$, $\phi_H^t(p) \in L_i$. Taking $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_{n-k}\}$, we get

$$\forall t \in [0, \varepsilon], \phi_H^t(p) \in \bigcap_{i=1}^{n-k} L_i \subset \theta^{-1}(\mathcal{F}_0(0)).$$

□

In the particular case of the standard coisotropic linear space $\mathcal{C}_0 \subset \mathbb{R}^{2n}$, we can prove a similar statement for all times. We will use it to prove Theorem 1.

Lemma 21. *Let $H \in C_{\text{Ham}}^0(\mathbb{R}^{2n}, \omega_0)$ be a compactly supported continuous Hamiltonian whose restriction to C_0 is a function of time. Then for every $t \in [0, +\infty)$, $\phi_H^t(0) \in \mathcal{F}_0(0)$.*

Proof. Assume this does not hold. Then $t_0 = \inf\{t > 0 \mid \phi_H^t(0) \notin \mathcal{F}_0(0)\}$ is a well-defined non-negative real number. Since $\mathcal{F}_0(0)$ is closed, it contains $p = \phi_H^{t_0}(0)$. According to Lemma 20 applied to the continuous Hamiltonian $K_t = H_{t+t_0}$, the point p , and the chart which is just the translation by $-p$, $\phi_K^t(p) \in \mathcal{F}_0(p)$ for every $t \in [0, \varepsilon]$ for some $\varepsilon > 0$. But since $\phi_K^t \phi_H^{t_0} = \phi_H^{t_0+t}$, we have $\phi_H^{t_0+t}(0) \in \mathcal{F}_0(0)$ for every $t \in [0, \varepsilon]$ which therefore contradicts the definition of t_0 . \square

The proof of Theorem 1 relies on the above lemma and the following characterization of coisotropic submanifolds and their characteristic foliations:

A submanifold is coisotropic if and only if the flow of every autonomous Hamiltonian constant on it preserves it. Moreover, the leaf through a point p is locally the union of the orbits of p under the flows of all such Hamiltonians.

The next lemma is based on this characterization.

Lemma 22. *Let C be a submanifold in a symplectic manifold (M, ω) . Assume that every point $p \in C$ admits an open neighborhood V such that any $H \in C_c^\infty(V)$, with $H|_C \equiv 0$, satisfies $\phi_H^t(p) \in C$ for every $t \in [0, +\infty)$. Then C is coisotropic.*

Moreover, for such a neighborhood V , there exists a smaller neighborhood $W \Subset V$ such that, the leaf $\mathcal{F}(p)$ of the characteristic foliation of C passing through p satisfies

$$W \cap \mathcal{F}(p) = W \cap \{\phi_H^t(p) \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_C \equiv 0\}.$$

Proof. Let $p \in C$ and let V be an open subset as in the statement of the lemma. Assume that C coincides locally with $f_1^{-1}(0) \cap \dots \cap f_k^{-1}(0)$ for some smooth functions f_1, \dots, f_k whose differentials are linearly independent at p . By multiplying by an appropriate cutoff function, we can assume that these functions are defined everywhere on M , have compact support in V , and vanish on C .

The Hamiltonian vector fields at p of f_1, \dots, f_k span $(T_p C)^\omega$, and by assumption belong to $T_p C$. Thus $(T_p C)^\omega \subset T_p C$ and C is coisotropic.

Now, since the characteristic leaves are preserved by smooth Hamiltonians constant on C , we have the inclusion

$$\mathcal{F}(p) \supset \{\phi_H^t(p) \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_C \equiv 0\}.$$

Conversely, consider the map

$$F: \mathbb{R}^k \rightarrow C, \quad (v_1, \dots, v_k) \mapsto \phi_{\sum_{i=1}^k v_i f_i}^1(p).$$

Since, $\sum_{i=1}^k v_i f_i$ is constant on C , its flow preserves the characteristics, hence F takes values in the characteristic leaf $\mathcal{F}(p)$ through p . The partial derivatives of F at 0 are $\partial_{v_i} F(0) = X_{f_i}(p)$ and in particular they are linearly

independent and $\text{span } T_p \mathcal{F}(p) = (T_p C)^\omega$. The inverse function theorem then shows that F is a diffeomorphism from a neighborhood of 0 to a neighborhood W of p in $\mathcal{F}(p)$. This shows

$$W \cap \mathcal{F}(p) \subset W \cap \{\phi_H^t(p) \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_C \equiv 0\}$$

and finishes the proof of Lemma 22. \square

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let C be a smooth coisotropic submanifold, and $\theta: U \rightarrow V$ be a symplectic homeomorphism. Assume $C' = \theta(U \cap C)$ is smooth. Let $p' \in C'$ and $p = \theta^{-1}(p')$. By passing to an appropriate Darboux chart around p , we may assume that $U \subset \mathbb{R}^{2n}$, $p = 0 \in \mathbb{R}^{2n}$, and $C = \mathcal{C}_0$. We are going to prove that any function $H \in C_c^\infty(V)$, with $H|_{C'} \equiv 0$, satisfies $\phi_H^t(p') \in C'$ for all $t \in [0, +\infty)$. According to Lemma 22, this will imply that C' is coisotropic.

Let H be such a function and consider the function $H \circ \theta$. It is compactly supported in U and can be extended to a continuous compactly supported function $K: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. Since H is smooth and θ is a symplectic homeomorphism, $K \in C_{\text{Ham}}^0(\mathbb{R}^{2n}, \omega_0)$. Since $K|_{\mathcal{C}_0} = 0$, Lemma 21 yields $\phi_K^t(0) \in \mathcal{F}_0(0)$ for any $t \geq 0$. Since K has support in U , we have

$$(9) \quad \forall t \geq 0, \phi_K^t(0) \in \mathcal{F}_0(0) \cap U \subset \mathcal{C}_0 \cap U.$$

Since $\phi_H^t = \theta \phi_K^t \theta^{-1}$, we deduce $\phi_H^t(p') \in C'$ as desired and hence that C' is coisotropic.

Denote \mathcal{F}' the characteristic foliation of C' . From (9), we deduce that for any $H \in C_c^\infty(V)$, $H|_{C'} \equiv 0$,

$$\forall t \geq 0, \phi_H^t(p') \in \theta(\mathcal{F}_0(0) \cap U).$$

Now according to Lemma 22, there exists a neighborhood $W \subset V$ such that

$$W \cap \mathcal{F}'(p') = W \cap \{\phi_H^t(p') \mid t \in [0, +\infty), H \in C_c^\infty(V), H|_{C'} \equiv 0\}.$$

Thus,

$$W \cap \mathcal{F}'(p') \subset W \cap \theta(\mathcal{F}_0(0)).$$

We get the reverse inclusion by switching the roles of C and C' , and we see that θ sends locally $\mathcal{F}_0(0)$ onto $\mathcal{F}'(p')$. \square

Proof of Corollary 4. If such a foliation exists it has to coincide with $\theta^{-1}(\mathcal{F}_0)$ on the domain of any C^0 -coisotropic chart φ . The only thing to check is that for any two C^0 -coisotropic charts $\theta_1: U_1 \rightarrow V_1$ and $\theta_2: U_2 \rightarrow V_2$, the foliations $\theta_1^{-1}(\mathcal{F}_0)$ and $\theta_2^{-1}(\mathcal{F}_0)$ coincide on $U_1 \cap U_2$. But this follows immediately from Theorem 1 applied to $C = \mathcal{C}_0$ and $\theta = \theta_1 \theta_2^{-1}: \theta_2(U_1 \cap U_2) \rightarrow \theta_1(U_1 \cap U_2)$. \square

Proof of Corollary 5. This is an obvious application of Theorem 1: We only need to apply Theorem 1 to every C^0 -coisotropic chart of our C^0 -coisotropic submanifold. \square

Proof of Theorem 6. Let $H \in C_{\text{Ham}}^0$ such that $H|_C$ is a function of time only and pick $p \in C$. For a contradiction, assume that for some $t > 0$, $\phi_H^t(p) \notin \mathcal{F}(p)$ and set $t_0 = \inf\{t > 0 \mid \phi_H^t(p) \notin \mathcal{F}(p)\}$. Then, consider the Hamiltonian $K_t = -H_{-t+t_0}$, so that $\phi_K^t = \phi_H^{-t+t_0}(\phi_H^{t_0})^{-1}$. Its restriction to C is also a function of time. Lemma 20 applied to K at the point $\phi_H^{t_0}(p)$ implies that for some small $t > 0$, $\phi_H^{t_0-t}(p) \in \mathcal{F}(\phi_H^{t_0}(p))$. But by definition of t_0 , we also have $\phi_H^{t_0-t}(p) \in \mathcal{F}(p)$, hence $\phi_H^{t_0}(p) \in \mathcal{F}(p)$. Now apply Lemma 20 again to H_{t+t_0} at the point $\phi_H^{t_0}(p)$. We get that for some $\varepsilon' > 0$ and all $t \in [t_0, t_0 + \varepsilon']$, $\phi_H^t(p) \in \mathcal{F}(p)$ which contradicts the definition of t_0 . Thus, $\phi_H^t(p) \in \mathcal{F}(p)$ and the direct implication of Theorem 6 follows.

We now prove the converse. Assume that the flow of $H \in C_{\text{Ham}}^0$ preserves each leaf of the characteristic foliation. We are going to show that for small times t , the function H_t is locally constant.

Let $p \in C$ and $\theta: U \rightarrow V$ be a C^0 -coisotropic chart around p , with $\theta(p) = 0$. For $\sigma > 0$ small enough, the set $\bigcap_{t \in [0, \sigma]} (\phi_H^t)^{-1}(U)$ contains p in its interior. Denote by U' this interior for some fixed σ . Similarly, for $s \in (0, \sigma]$ small enough, $\bigcap_{t \in [0, s]} \phi_H^t(U')$ contains p in its interior. Let U'' be an open neighborhood of p contained in this interior, and with the property that $\theta(U'')$ is convex. Let q be any other point in U'' and Λ be a linear Lagrangian subspace included in \mathcal{C}_0 , containing $\theta(q)$ and the standard leaf $\mathcal{F}_0(0)$. Such a subspace is the union of the leaves $F_0(x)$ for all $x \in \Lambda$.

Now, consider the C^0 -Lagrangian $L = \theta^{-1}(\Lambda \cap V)$. Let $\mathcal{U} = L \cap U'$ and $\mathcal{V} = L \cap U''$. By construction, $q \in \mathcal{V}$. By assumption $\phi_H^t(\mathcal{U}) \subset L$ for all $t \in [0, s]$. We may apply Proposition 18 to L and the continuous Hamiltonian $K_t(x) = sH_{st}(x)$ which generates the homotopy ϕ_H^{st} . We get that for any $t \in [0, 1]$, K_t is locally constant on \mathcal{V} . Equivalently, for any $t \in [0, s]$, H_t is locally constant on \mathcal{V} . Now since $\theta(U'')$ is convex and Λ is linear, $\theta(U'') \cap \Lambda$ is connected. It follows that \mathcal{V} is also connected and therefore $H_t(p) = H_t(q)$. To summarize, we proved that for t small enough, H_t is constant on U'' .

Since C is assumed to be connected, this means that H_t is constant for t small enough. However, the argument we followed for times close to 0 applies for any other initial time. Thus, H_t must be constant for any time. \square

APPENDIX A. A SMOOTH C^0 -LAGRANGIAN IS LAGRANGIAN

In this section we provide relatively simple proof of Theorems 1 and 6 in an enlightening and important special case. We suppose that $M = T^*L$ equipped with its canonical symplectic structure for some closed smooth manifold L . Denote by θ a symplectic homeomorphism of T^*L . And let $L' = \theta(L_0)$, where L_0 denotes the 0-section of T^*L .

Below, we will prove Theorems 1 and 6 for the C^0 -Lagrangian L' . In this setting the mentioned theorems state the following:

Theorem 23. *If L' is smooth, then it is Lagrangian.*

Theorem 24. *Let $H \in C_{\text{Ham}}^0$ with induced hameotopy ϕ_H . The restriction of H to L' is a function of time if and only if ϕ_H preserves L' .*

We believe that the above special cases provide the reader with the opportunity to get an idea of the proofs of Theorems 1 and 6 without having to go through the technical details of Sections 4 and 5.

We will first show that Theorem 23 follows from Theorem 24. In order to do so we will need the following dynamical characterizations of isotropic and coisotropic submanifolds, respectively.

Lemma 25. *Let I denote a (smooth) submanifold of a symplectic manifold (M, ω) . The following are equivalent:*

- I is isotropic,
- For every smooth Hamiltonian H , if ϕ_H preserves I , then $H|_I$ is a function of time only.

Lemma 26. *Let C denote a (smooth) submanifold of a symplectic manifold (M, ω) . The following are equivalent:*

- C is coisotropic,
- For every smooth Hamiltonian H , if $H|_C$ is a function of time only, then ϕ_H preserves C .

We leave the proofs of the above lemmas, which follow from symplectic linear algebra, to the reader. In the proof of Theorem 1, we use Lemma 22 which is a variation of the second of the above two lemmas.

Proof of Theorem 23. Each of Lemmas 25 and 26 gives a different proof. We provide both proofs here.

First proof: Suppose that H is any smooth Hamiltonian whose flow ϕ_H preserves L' . Then $H \circ \theta \in C_{\text{Ham}}^0$ and its flow, $\theta^{-1}\phi_H^t\theta$, preserves L_0 . It follows from Theorem 24 that the restriction of $H \circ \theta$ to L_0 is a function of time only. Therefore, $H|_{L'}$ depends on time only, and so using Lemma 25 we conclude that L' is isotropic.

Second proof: Suppose that H is any smooth Hamiltonian whose restriction to L' is a function of time only. Then $H \circ \theta \in C_{\text{Ham}}^0$ and its restriction to L_0 depends only on time. It follows from Theorem 24 that the flow of $H \circ \theta$, which is $\theta^{-1}\phi_H^t\theta$, preserves L_0 and so the flow of H preserves L' . Using Lemma 26 we conclude that L' is coisotropic. \square

Proof of Theorem 24. By replacing H with $H \circ \theta$ we may prove the statement for the (smooth) Lagrangian L_0 , rather than for L' . Here, we use the fact that $\phi_{H \circ \theta}^t = \theta^{-1}\phi_H^t\theta$. Hence, we will prove that *the restriction of $H \in C_{\text{Ham}}^0$ to L_0 is a function of time if and only if ϕ_H preserves L_0 .*

To prove the direct implication suppose that $H_t|_{L_0} = c(t)$, where $c(t)$ is a function of time only. By replacing H_t with $H_t - c(t)$ and then cutting off at infinity, we may assume that $H_t|_{L_0} = 0$ for all t . For a contradiction assume that ϕ_H does not preserve L_0 and define $t_0 = \inf\{t \mid \phi_H^t(L_0) \not\subset L_0\}$.

Pick $\delta > 0$ such that $\phi_H^{t_0+\delta}(L_0) \not\subset L_0$. By a time reparametrization we may assume that $t_0 = 0$ and $\delta = 1$, that is $\phi_H^1(L_0) \not\subset L_0$.

Since $H \in C_{\text{Ham}}^0$ there exists a sequence of smooth Hamiltonians $H_i: [0, 1] \times M \rightarrow \mathbb{R}$ such that H_i converges uniformly to H and ϕ_{H_i} converges to ϕ_H in C^0 -topology.

Because $\phi_H^1(L_0) \not\subset L_0$, there exists a ball B such that $B \cap L_0 \neq \emptyset$ and $\phi_H^1(L_0) \cap B = \emptyset$. It follows that $\phi_{H_i}^1(L_0) \cap B = \emptyset$ for large i . And so,

$$\gamma(\phi_{H_i}^1(L_0), L_0) \geq c_{\text{LR}}(B; L_0) > 0.$$

Inequality (1) from Section 2.2 implies that

$$\max_{t \in [0, 1]} (\text{osc}(H_i(t, \cdot)|_{[0, 1] \times L_0})) \geq c_{\text{LR}}(B; L_0),$$

contradicting the fact that $H|_{L_0} = 0$. We conclude that ϕ_H preserves L_0 .

Next, to prove the converse implication suppose that ϕ_H preserves L_0 . Let B denote any open ball intersecting L_0 , U a small open neighborhood of \overline{B} and ψ be any symplectomorphism supported in B and preserving L_0 . Next, we pick $\varepsilon > 0$ such that $\phi_H^t(B), (\phi_H^t)^{-1}(B) \Subset U$ for all $t \in [0, \varepsilon]$. By a reparametrization in time, we may assume that $\phi_H^t(B), (\phi_H^t)^{-1}(B) \Subset U$ for all $t \in [0, 1]$.

Consider the C^0 -Hamiltonian $G = (H \circ \psi - H) \circ \phi_H$. We will show that $G|_{L_0} = 0$. Since B , and ψ are chosen arbitrarily, this will show that H is a function of time. The support of G is included in $\cup_{t \in [0, 1]} (\phi_H^t)^{-1}(B) \subset U$, and moreover, its flow is $(\phi_H^t)^{-1} \psi^{-1} \phi_H^t \psi$. Because ψ and ϕ_H preserve L_0 the flow of G also preserves L_0 .

Since $G \in C_{\text{Ham}}^0$ there exist smooth Hamiltonians G_i such that $\{G_i\}$ converges uniformly to G and $\{\phi_{G_i}\}$ converges to ϕ_G . Furthermore, we can require that all G_i s are supported in U . This can be achieved by picking a corresponding sequence of smooth Hamiltonians H_i for H and defining $G_i = (H_i \circ \psi - H_i) \circ \phi_{H_i}$. For large i , G_i is supported in U .

Fix a small $r > 0$. Because $\phi_G^t(L_0) = L_0$ for any $t \in [0, 1]$, for sufficiently large i we have $\phi_{G_i}^t(L_0) \subset T_r^* L_0$. Furthermore, the Hamiltonians G_i are all supported in U and hence we can apply Lemma 10 and conclude that $\gamma(\phi_{G_i}^1(L_0), L_0) \leq Cr$, i.e $\gamma(\phi_{G_i}^1(L_0), L_0) \rightarrow 0$. Of course, the same reasoning yields $\gamma(\phi_{G_i}^t(L_0), L_0) \rightarrow 0$ for all $t \in [0, 1]$. Then, Proposition 19 implies that $G|_{L_0} = c(t)$. Since it has support in U , we conclude that $G|_{L_0} = 0$. \square

APPENDIX B. GRAPHS OF CLOSED C^0 1-FORMS ARE C^0 -LAGRANGIANS

The goal of this appendix is to provide a family of non trivial examples of C^0 -Lagrangians by proving the following statement mentioned in the introduction.

Proposition 27. *Let α be a C^0 1-form on a smooth manifold N which is closed in the sense of distributions. Then its graph, $\text{graph}(\alpha) \subset T^*N$, is a C^0 -Lagrangian.*

Proof. Since the statement is local, it is sufficient to prove it when N is an open set of \mathbb{R}^n . Then α can be written as $\alpha = \sum_{i=1}^n p_i(x) dx_i$, where x_1, \dots, x_n are the canonical coordinates in \mathbb{R}^n and p_1, \dots, p_n continuous functions on N . The fact that α is closed is equivalent to the equations

$$(10) \quad \forall i, j \in \{1, \dots, n\}, \quad \partial_j p_i = \partial_i p_j,$$

where $\partial_i p_j$ is the i -th partial derivative of p_j in the sense of distributions.

We use convolution to approximate α . To that end, take a compactly supported function ρ such that $\rho \geq 0$, and $\int_N \rho(x) dx = 1$ and set $\rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho(\varepsilon x)$ for every $\varepsilon > 0$. For any continuous function f on N , the functions

$$f * \rho_\varepsilon(x) = \int_N f(y) \rho_\varepsilon(x - y) dy$$

are well-defined on any compact subset of N for ε small enough. Moreover, for any ε , $f * \rho_\varepsilon$ is smooth, converges locally uniformly to f as ε goes to 0, its differential satisfies $d(f * \rho_\varepsilon) = (df) * \rho_\varepsilon$ and converges in the sense of distributions to df .

Let $U \Subset N$ be an open subset of N . Then, for ε small enough,

$$\alpha_\varepsilon = \sum_{i=1}^n p_i * \rho_\varepsilon dx_i$$

is a well-defined 1-form on U . It satisfies Equations (10) and thus is closed. Moreover, it converges uniformly to α on U .

Now let ϕ_ε be the family of symplectic diffeomorphisms of T^*U defined by $\phi_\varepsilon(x, p) = (x, p + \alpha_\varepsilon(x))$. They converge uniformly on U to the symplectic homeomorphism $\phi: T^*U \rightarrow T^*U$, $(x, p) \mapsto (x, p + \alpha(x))$ and $\text{graph}(\alpha)$ restricted to T^*U is $\phi(U)$. This shows that $\text{graph}(\alpha)$ is locally the image of a smooth Lagrangian by a symplectic homeomorphism. \square

REFERENCES

- [1] J.-F. Barraud and O. Cornea. Lagrangian intersections and the Serre spectral sequence. *Ann. of Math. (2)*, 166(3):657–722, 2007.
- [2] M. S. Borman and M. McLean. The width of a Lagrangian and Floer–Hofer capacities. (in preparation).
- [3] L. Buhovsky and S. Seyfaddini. Uniqueness of generating Hamiltonians for topological Hamiltonian flows. *J. Symplectic Geom.*, 11(1):37–52, 2013.
- [4] F. Charette. A geometric refinement of a theorem of Chekanov. *J. Symplectic Geom.*, 10(3):475–491, 2012.
- [5] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
- [6] U. Frauenfelder, V. L. Ginzburg, and F. Schlenk. Energy capacity inequalities via an action selector. In *Geometry, spectral theory, groups, and dynamics*, volume 387 of *Contemp. Math.*, pages 129–152. Amer. Math. Soc., Providence, RI, 2005.
- [7] V. L. Ginzburg. Coisotropic intersections. *Duke Math. J.*, 140(1):111–163, 2007.
- [8] M. J. Gotay. On coisotropic imbeddings of presymplectic manifolds. *Proc. Amer. Math. Soc.*, 84(1):111–114, 1982.

- [9] M. Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.
- [10] V. Humilière, R. Leclercq, and S. Seyfaddini. New energy-capacity-type inequalities and uniqueness of continuous Hamiltonians. *Comment. Math. Helv.*, to appear (arXiv:1209.2134).
- [11] F. Laudenbach and J.-C. Sikorav. Hamiltonian disjunction and limits of Lagrangian submanifolds. *Internat. Math. Res. Notices*, (4):161 ff., approx. 8 pp. (electronic), 1994.
- [12] R. Leclercq. Spectral invariants in Lagrangian Floer theory. *J. Mod. Dyn.*, 2(2):249–286, 2008.
- [13] R. Leclercq and F. Zapolsky. Spectral invariants for monotone Lagrangian submanifolds. (in preparation).
- [14] P. Libermann and C.-M. Marle. *Symplectic geometry and analytical mechanics*, volume 35 of *Mathematics and its Applications*. D. Reidel Publishing Co., Dordrecht, 1987. Translated from the French by Bertram Eugene Schwarzbach.
- [15] S. Lisi and A. Rieser. Coisotropic Hofer–Zehnder capacities. (in preparation).
- [16] D. Milinković. On equivalence of two constructions of invariants of lagrangian submanifolds. *Pacific J. of Math.*, 195:371–415, 2000.
- [17] A. Monzner, N. Vichery, and F. Zapolsky. Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization. *J. Mod. Dyn.*, 6(2):205–249, 2012.
- [18] Y.-G. Oh. Symplectic topology as the geometry of action functional. I. Relative Floer theory on the cotangent bundle. *J. Differential Geom.*, 46(3):499–577, 1997.
- [19] Y.-G. Oh. Locality of continuous Hamiltonian flows and Lagrangian intersections with the conormal of open subsets. *J. Gökova Geom. Topol. GGT*, 1:1–32, 2007.
- [20] Y.-G. Oh. Geometry of generating functions and Lagrangian spectral invariants. *ArXiv:1206.4788*, June 2012.
- [21] Y.-G. Oh and S. Müller. The group of Hamiltonian homeomorphisms and C^0 -symplectic topology. *J. Symplectic Geom.*, 5(2):167–219, 2007.
- [22] E. Opshtein. C^0 -rigidity of characteristics in symplectic geometry. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):857–864, 2009.
- [23] M. Usher. The sharp energy-capacity inequality. *Commun. Contemp. Math.*, 12(3):457–473, 2010.
- [24] C. Viterbo. Symplectic topology as the geometry of generating functions. *Math. Annalen*, 292:685–710, 1992.
- [25] C. Viterbo. On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows. *Int. Math. Res. Not.*, pages Art. ID 34028, 9, 2006.
- [26] C. Viterbo. Symplectic homogenization. *ArXiv:0801.0206*, Dec. 2008.
- [27] A. Weinstein. Symplectic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 5(1):1–13, 1981.

VH: INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UNIVERSITÉ PIERRE ET MARIE CURIE, 4 PLACE JUSSIEU, 75005 PARIS, FRANCE
E-mail address: humi@math.jussieu.fr

RL: UNIVERSITÉ PARIS-SUD, DÉPARTEMENT DE MATHÉMATIQUES, BAT. 425, 91405 ORSAY CEDEX, FRANCE
E-mail address: remi.leclercq@math.u-psud.fr

SS: DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS DE L'ÉCOLE NORMALE SUPÉRIEURE, 45 RUE D'ULM, F 75230 PARIS CEDEX 05
E-mail address: sobhan.seyfaddini@ens.fr